the subject, and to Hart's discovery of a five-bar linkage which does the same work as Peaucellier's of seven. Henceforth Peaucellier's Cell and Hart's Contraparallelogram will take their place in our text-books of geometry, and straight lines can be drawn without begging the question by assuming first a straight edge or ruler as does Euclid. Thus Kempe's charming book, *How to draw a straight line*, is a direct outcome of Tchebychev's sketch for Sylvester, to whom, in parting, he used the characteristic words: "Take to kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space." As might perhaps have been expected, the immortal Lobachevsky found in his compatriot a devoted admirer. Not only was Tchebychev an active member of the Committee of the Lobachevsky-fund, but he took the deepest interest in all connected with the spread of the profound ideas typified in the Non-Euclidean geometry.

Knowing this, Vasiliev in his last letter asked that a copy of my translation of his Address on Lobachevsky be forwarded to the great man. His active participation in scientific assemblies is also worthy of note; for example at the 'Congres de l'Association francaise pour l' avancement des sciences, a Lyon' he read two interesting papers, *Sur les valeurs limites des integrales*, and *Sur les quadratures*, both afterwards published in *Liouville's Journal*.

LAGRANGE'S GENERALIZED EQUATIONS OF MOTION.

By F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Lagrange's celebrated equations of motion, as given in his *Mecanique Analytique*, consist of *a transformation* from Cartesian to generalized co-ordinates, *of the indeterminate equation of motion.* In order to avoid the unnecessary complication incident to the introduction of such indeterminate variations, as δx, δy, δz, etc., in the derivation of these equations of motion, we must have recourse to the Cartesian equations of the unconstrained motion of particles.

Assume $(x_1,y_1,z_1),\ldots,(x_n,y_n,z_n)$ as the rectilineal-rectangular co-ordinates of the material particles $m_1,\ldots,m_n$; also, assume $(X_1,Y_1,Z_1),\ldots,(X_n,Y_n,Z_n)$ as the force components of the particles. Make $(\psi_1,\phi_1,\theta_1),\ldots,(\psi_n,\phi_n,\theta_n)$ the generalized co-ordinates of the *respective* particles at any instant of time; that is, let ψ_1,ϕ_1,θ_1, be regarded as determinate functions of x_1,y_1,z_1, respectively,—or vice versa. This reciprocal determinateness is to be a characteristic of all the material particles. The velocity-components in

Cartesian co-ordinates are $(dx_1 / dt, dy_1 / dt, dz_1 / dt)$, etc.; while the corresponding generalized velocity-components are $(d\psi_1 / dt, d\phi_1 / dt, d\theta_1 / dt)$, etc. Similarly the acceleration-components in Cartesiàn co-ordinates are $(d^2x_1 / dt^2, d^2y_1 / dt^2, d^2z_1 / dt^2)$, etc.; while the corresponding generalized acceleration-components are $(d^2\psi_1 / dt^2, d^2\phi_1 / dt^2, d^2\theta_1 / dt^2)$, etc.

From well-known principles of the differential calculus,

$$\frac{dx_1}{dt} = \frac{dx_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dx_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dx_1}{d\theta_1}\cdot\frac{d\theta_1}{dt} \ldots\ldots (A_1),$$

$$\frac{dy_1}{dt} = \frac{dy_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dy_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dy_1}{d\theta_1}\cdot\frac{d\theta_1}{dt} \ldots\ldots (A_2),$$

$$\frac{dz_1}{dt} = \frac{dz_1}{d\psi_1}\cdot\frac{d\psi_1}{dt} + \frac{dz_1}{d\phi_1}\cdot\frac{d\phi_1}{dt} + \frac{dz_1}{d\theta_1}\cdot\frac{d\theta_1}{dt} \ldots\ldots (A_3).$$

Since the kinetic energy of the particle m_1 represented by the Cartesian co-ordinates (x_1, y_1, z_1) becomes

$$K_1 = \tfrac{1}{2}m_1[(dx_1 / dt)^2 + (dy_1 / dt)^2 + (dz_1 / dt)^2] \ldots\ldots (a),$$

it is obvious [substituting the square of (A_1), of (A_2), and of (A_3), in (a)] that the corresponding kinetic energy expressed in terms of generalized co-ordinates may be represented (according to the usual system of functional notation) by

$$T_1 = \tfrac{1}{2}\Big[(\psi_1,\psi_1)\Big(\frac{d\psi_1}{dt}\Big)^2 + (\phi_1,\phi_1)\Big(\frac{d\phi_1}{dt}\Big)^2 + (\theta_1,\theta_1)\Big(\frac{d\theta_1}{dt}\Big)^2$$

$$+ 2(\psi_1,\phi_1)\Big(\frac{d\psi_1}{dt}\cdot\frac{d\phi_1}{dt}\Big) + 2(\phi_1,\theta_1)\Big(\frac{d\phi_1}{dt}\cdot\frac{d\theta_1}{dt}\Big) + 2(\theta_1,\psi_1)\Big(\frac{d\theta_1}{dt}\cdot\frac{d\psi_1}{dt}\Big)\Big] \ldots (a').$$

The *coefficients* in (a'), *inclosed by the smaller parentheses*, are homogeneous functions of the co-odinates. These coefficients are determinable from system-conditions. With respect to these coefficients, the necessary condition is that they must give a finite and positive value of T_1, for whatever values may be assigned to the variables.

If $(\delta x_1, \delta y_1, \delta z_1), \ldots, (\delta x_n, \delta y_n, \delta z_n)$ represent the components of any infinitely small motions possible without breaking the conditions of the material system of which $m_1, \ldots, m_n$ are the component particles; then *the work done by the forces* whose components have already been specified, *upon the particle* m_1, becomes $W_1 = X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 \ldots\ldots (b)$.

In order to deduce the expression for W_1 in terms of generalized co-ordinates, we have from the differential calculus:

$$\delta x_1 = \frac{dx_1}{d\psi_1}\delta\psi_1 + \frac{dx_1}{d\phi_1}\delta\phi_1 + \frac{dx_1}{d\theta_1}\delta\theta_1 \ldots\ldots (B_1),$$

$$\delta y_1 = \frac{dy_1}{d\psi_1}\delta\psi_1 + \frac{dy_1}{d\phi_1}\delta\phi_1 + \frac{dy_1}{d\theta_1}\delta\theta_1 \ldots\ldots (B_2),$$

$$\delta z_1 = \frac{dz_1}{d\psi_1} + \frac{dz_1}{d\phi_1}\delta\phi_1 + \frac{dz_1}{d\theta_1}\delta\theta_1 \ldots\ldots (B_3).$$

Multiplying (B_1) by X_1, (B_2) by Y_1, (B_3) by Z_1, and adding products, we have

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 = \left(X_1\frac{dx_1}{d\psi_1} + Y_1\frac{dy_1}{d\psi_1} + Z_1\frac{dz_1}{d\psi_1}\right)\delta\psi_1$$

$$\left(X_1\frac{dx_1}{d\phi_1} + Y_1\frac{dy_1}{d\phi_1} + Z_1\frac{dz_1}{d\phi_1}\right)\delta\phi_1 + \left(X_1\frac{dx_1}{d\theta_1} + Y_1\frac{dy_1}{d\theta_1} + Z_1\frac{dz_1}{d\theta_1}\right)\delta\theta_1,$$

$= \Psi_1\delta\psi_1 + \Phi_1\delta\phi_1 + \Theta_1\delta\theta_1 \ldots (d_1)$, in which the coefficients of the indeterminate variations are the generalized force-components with respect to the particle m_1. According to D'Alembert's principle, the Cartesian *Equations of motion* in the order of the particles specified, become

$$X_1 = m_1\frac{d^2x_1}{dt^2},\ Y_1 = m_1\frac{d^2y_1}{dt^2},\ Z_1 = m_1\frac{d^2z_1}{dt^2}, \ldots,$$

$$X_n = m_n\frac{d^2x_n}{dt^2},\ Y_n = m_n\frac{d^2y_n}{dt^2}, Z_n = m_n\frac{d^2z_n}{dt^2}.$$

Multiplying these equations, respectively, by

$\frac{dx_1}{d\psi_1}, \frac{dy_1}{d\psi_1}, \frac{dz_1}{d\psi_1}, \ldots, \frac{dx_n}{d\psi_n}, \frac{dy_n}{d\psi_n}, \frac{dz_n}{d\psi_n}$; and then adding the products, we have the equations,

$$\Psi = m_1\left(\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\psi_1} + \frac{d^2y_1}{dt^2}\cdot\frac{dy_1}{d\psi_1} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\psi_1}\right) + \ldots$$

$$+ m_n\left(\frac{d^2x_n}{dt^2}\cdot\frac{dx_n}{d\psi_n} + \frac{d^2y_n}{dt^2}\cdot\frac{dy_n}{d\psi_n} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\psi_n}\right) \ldots (c_1).$$

After performing similar operations, we have the equations:

$$\Phi = m_1\left(\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\phi_1} + \frac{d^2y_1}{dt^2}\cdot\frac{dy_1}{d\phi_1} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\phi_1}\right) + \ldots$$

$$+ m_n\left(\frac{d^2x_n}{dt^2}\cdot\frac{dx_n}{d\phi_n} + \frac{d^2y_n}{dt^2}\cdot\frac{dy_n}{d\phi_n} + \frac{d^2z_n}{dt^2}\cdot\frac{dz_n}{d\phi_n}\right) \ldots (c_2),$$

and $\Theta = m_1\left(\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\theta_1} + \frac{d^2y_1}{dt^2}\cdot\frac{dy_1}{d\theta_1} + \frac{d^2z_1}{dt^2}\cdot\frac{dz_1}{d\theta_1}\right) + \ldots$

$$+ m_n\left(\frac{d^2x_n}{dt^2}\cdot\frac{dx_n}{d\theta_n} + \frac{d^2y_n}{dt^2}\cdot\frac{dy_n}{d\theta_n} + \frac{d^2z_n}{dt^2}\cdot\frac{dz_n}{d\theta_n}\right) \ldots (c_3).$$

Taken in its utmost generality; that is, in case of *isolated* and *con-*

servative Material-systems, (a) becomes $K=\Sigma\frac{1}{2}m[(dx \,/\, dt)^2+(dy \,/\, dt)^2 + (dz \,/\, dt)^2]\ldots(a_1)$; also, from (a'), in case of analogous systems,

$$T=\frac{1}{2}\Big[(\psi,\psi)\Big(\frac{d\psi}{dt}\Big)^2+(\phi,\phi)\Big(\frac{d\phi}{dt}\Big)^2+(\theta,\theta)\Big(\frac{d\theta}{dt}\Big)^2$$

$$+2(\psi,\phi)\Big(\frac{d\psi}{dt}\cdot\frac{d\phi}{dt}\Big)+2(\phi,\theta)\Big(\frac{d\phi}{dt}\cdot\frac{d\theta}{dt}\Big)+2(\theta,\psi)\Big(\frac{d\theta}{dt}\cdot\frac{d\psi}{dt}\Big)\Big]\ldots(a'_1).$$

Supposing $dx_1 \,/\, dt$ to be a function of the Cartesian co-ordinates, and also a function of the generalized velocity-components, we have from the differential calculus:

$$\frac{d^2x_1}{dt^2}\cdot\frac{dx_1}{d\psi_1}=\frac{d}{dt}\Big(\frac{dx_1}{dt}\cdot\frac{dx_1}{d\psi_1}\Big)-\frac{dx_1}{dt}\cdot\frac{d}{dt}\Big(\frac{dx_1}{d\psi_1}\Big),$$

$$=\frac{d}{dt}\Big[\frac{dx_1}{dt}\Big(\frac{dx_1}{dt}\Big/\frac{d\psi_1}{dt}\Big)\Big]-\frac{dx_1}{dt}\Big(\frac{dx_1}{dt}\Big/d\psi_1\Big),$$

$$=\frac{d}{dt}\Big[\frac{1}{2}\text{ of }\frac{d[(dx_1 \,/\, dt)^2]}{d(d\psi_1 \,/\, dt)}\Big]-\frac{1}{2}\text{ of }\frac{d[(dx_1 \,/\, dt)^2]}{d\psi_1}\ldots(c).$$

For *every* term of (c_1), (c_2), and (c_3) *the proper expression* can be written by analogy, from the right-hand member of (c).

Transforming (c_1), (c_2), and (c_3), by means of (c) *generally applied* and then using T for the kinetic energy of the system, we have

$$\frac{d}{dt}\Big(dT\Big/\frac{d\psi}{dt}\Big)-\frac{dT}{d\psi}=\Psi,\ \frac{d}{dt}\Big(dT\Big/\frac{d\phi}{dt}\Big)-\frac{dT}{d\phi}=\Phi,$$

and $\frac{d}{dt}\Big(dT\Big/\frac{d\theta}{dt}\Big)-\frac{dT}{d\theta}=\Theta,$

which are Lagrange's equations of motion in terms of generalized co-ordinates.

With respect to any isolated and conservative material-system, the generalization of (b_1) gives $\Sigma(X\delta x+Y\delta y+Z\delta z)=\Psi\delta\psi+\Phi\delta\phi+\Theta\delta\theta\ldots(d)$.

The potential energy of a conservative system is a function of the co-ordinates by which the different positions of the various parts of such a system are specified. With reference to the configuration which an isolated and conservative material-system has at any instant, the potential energy represents the amount of work required to bring the system to that configuration against its mutual forces during the passage of the system from any one chosen configuration to the configuration referred to at the time. Hence if an aggregation of moving particles constitutes an isolated and conservative material system whose potential energy in the configuration specified by the Cartesian co-ordinates x, y, z) is represented by V, we must have $\delta V=-\Sigma(X\delta x+Y\delta y+Z\delta z)\ldots(e)$.

From (d) and (e), $\Psi\delta\psi+\Phi\delta\phi+\Theta\delta\theta=-\delta V\ldots(f)$.

$$\therefore\quad \Psi=-\frac{dV}{d\psi},\ \Phi=-\frac{dV}{d\phi},\ \text{and}\ \Theta=-\frac{dV}{d\theta}.$$

Therefore the Lagrangian equations of motion may be written:

$$\frac{d}{dt}\left(dT\Big/\frac{d\psi}{dt}\right)=\frac{dT}{d\psi}-\frac{dV}{d\psi},\ \frac{d}{dt}\left(dT\Big/\frac{d\phi}{dt}\right)=\frac{dT}{d\phi}-\frac{dV}{d\phi},$$

$$\text{and}\ \frac{d}{dt}\left(dT\Big/\frac{d\theta}{dt}\right)=\frac{dT}{d\theta}-\frac{dV}{d\theta}.$$

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from the February Number].

PPOPOSITION XIV. *The hypothesis of obtuse angle is inconsistent with Euclid's assumption: Two straight lines cannot enclose a space.*

Proof. From the hypthesis of obtuse angle, assumed as true, [and the first 28 propositions of Euclid], we have now deduced the truth of Euclid's Postulatum; that two straights will meet each other in some point toward those parts, toward which a certain straight, cutting them, makes two internal angles, of whatever kind, less than two right angles.

But this Postulatum holding good, on which Euclid supports himself after the twenty-eighth proposition of his first book, it is manifest to all Geometers, that the hypothesis of right angle alone is true, nor any place left for the hypothesis of obtuse angle. Therefore the hypothesis of obtuse angle is inconsistent with Euclid's assumption. Quod erat demonstrandum.

Otherwise, and more immediately.

Since from the hypothesis of obtuse angle we have demonstrated (P. IX.) that two (fig. 11.) acute angles of the triangle APX, right-angled at P, are greater than one right angle; it follows that an acute angle PAD may be assumed such, that together with the aforesaid two acute angles it makes up two right angles. But then the straight AD must (by the preceding proposition,